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Mathematics News Letter

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To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

VOL. 5 BATON ROUGE, LA., SEPTEMBER, 1930 NO. 1

KEEP OUT OF THE RUTS!

It matters not whether the mathematics to be taught is the mathematics of the grade, the high school or the college, the teaching of it by a mere "rote" method is NOT the best teaching. Because a certain way of administering a mathematics lesson last year brought fairly good results is not a justification for using the same method without change this year. As long as the personnel of student bodies change so long must the instructor be on the alert to decide needed modifications of previous teaching procedure. To be thus on the alert is not an easy thing to do. On the contrary, the very easiest thing to do is to so deepen the rut of method by constant unvarying repetition of it that the instructor can be half-asleep and yet use it—a deplorable state of things.

Staying in a teaching rut is not only death to all habits of rationalizing the lesson material for the student but, in addition, it furnishes but slight inspiration to him to maintain an ever thoughtful attitude to his mathematics—an attitude of investigation, of inquiry, and even of research. —*S. T. S.*

MASS TEACHING VERSUS INDIVIDUAL LEARNING

We recently heard one of the greatest of living mathematicians say that absolute self-guidance in mathematical study is now no longer possible. For this reason no one may now properly be called a "self-made" mathematician. The truth of the statement holds in less degree for the student of secondary mathematics, in greater degree for the student of college mathematics, in greatest degree for the graduate mathematical student.

But with an unduly large class of freshmen, there is great danger of too much guidance by the instructor. This is true because the lecture mode of instruction, in case the class is large, appears the mode most favored by tradition and practice. "Lecturing" mathematics into the freshman head is by far the poorest way to bring him to an acquaintance with the type and measure of his own mathematical ability. Within limits the instructor's plan should be, rather, to find a way to **maximize the individual student's interest in guiding himself.** It may be reasonably assumed that there are some things which he can learn by himself. The psychological opportunity of the instructor arrives when the student's first apparent insuperable difficulty is developed. At such moment a teacher-student conference to "lay low" the difficulty is always tellingly effective. The student interest will always be at "white heat" at such time.

Now such a fundamental method—not a method of instruction, but a method of interesting the student in learning by himself as much as possible—is to a very great degree independent of the size of the class. Given a room with sufficient seating capacity, fifty students can just as easily "sit and study and LEARN" in it as can a mere dozen. Of course there will be more individual conferences between teacher and student in the former case than in the latter, but the difference will add but little to the administrative troubles, and this little will be more than counterbalanced by the larger increase of power and interest of the students in their course work. Moreover, the **individual contact** between the teacher and the individual student has been put over with a 100% bang. —S. T. S.

*CHILDREN'S DIFFICULTIES WITH REASONING PROBLEMS

By LOLITA ROCA,
Capdau School, New Orleans, La.

The importance and the general practical aims of arithmetic have been so long recognized and are so obvious that little need be said about them. It may be said, however, that the modern conception of elementary arithmetic is that it is wholly practical and utilitarian, and that it has no other purpose than to prepare the pupil to understand and to solve the various problems that are a part of our daily life.

Problem solving necessarily holds an important part in the arithmetic curriculum.

However, training children to solve arithmetic problems is one of the hardest and most discouraging tasks of the teacher. Are problems wrong, or are our methods wrong, or both, or neither? The pupils seem to have a way of doing the wrong thing, of simply juggling the numbers, that is most exasperating. Just where do the children's difficulties in reasoning problems lie?

The difficulties that are most notable outside of those due to individual differences are:

I. *Unfamiliarity with the situation involved in the problem.*

This question was studied by the Committee of Seven of the Superintendents' and Principals' Association of Northern Illinois. Their experiment was conducted with more than 1,000 children. Ten problems of the same mathematical difficulty were given—five involving familiar situations and five involving unfamiliar ones. Two of the ten problems were:

(1) A farmer bought a new wire fence for his field. The shape of the field is shown in the following drawing with the

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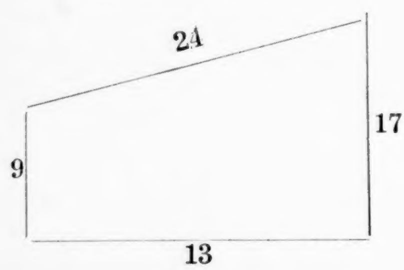
*Read at the New Orleans Public School Teacher's Institute, September 9, 1930.

length of each side in rods marked. The fencing cost \$5 a rod. What did the whole fence cost?



(2) The members of the Dramatic Club of the high school voted to tax themselves \$3 each to buy a new curtain for the stage. In the club there are 18 Seniors, 15 Juniors, 9 Sophomores, and 26 Freshmen. How much money can they raise?

The results of the experiment show that unfamiliarity with materials and situations in problems is a significant factor in causing difficulty with reasoning problems.

Remedies: The textbook problems containing unfamiliar situations could be substituted by real-life problems. These problems can be based on:

(1) Things the pupils are doing in school. (Reading, language, history, geography and other subjects of the curriculum may be made the basis for problems. Keeping scores in games, laying out of a playground or ball diamond, planning of school entertainments, etc., lead to interesting problems.)

(2) Things the pupils are doing outside of school. (Problems gathered by the pupils from the home and brought to the classroom. One such problem might occupy the attention of the class if not every day, at least once a week, the individual pupils presenting their problems in turn throughout the term. In one class each pupil was asked to bring one real-life problem involving addition, subtraction, multiplication or division of whole numbers, fractions, or decimals which the child had had occasion to solve. As a result, some problems were made up by the pupils for the occasion, a few were copied by the pupils from other textbooks, but the majority were typical of those problems

found in everyday occupations. In many cases the handwriting clearly showed that the child's parents were the writers of the problems, this fact indicating genuine interest and readiness to assist.)

(3) Community activities which either are familiar or can easily be made familiar to the pupil. (In a city we have local industries, improvements of streets, sales at local stores, building of houses, etc.)

II. *Lack of vocabulary and lack of knowledge of the meaning of words common to arithmetic.*

In reference to the former this might be given as an example:

Teacher: Why did you subtract?
Pupil: Because it says "increase."
Teacher: What does "increase" mean?
Pupil: Less.

In reference to the latter, pupils often misinterpret such words as per, lb. (for pound), etc.

Remedies: When words in the textbook problems are not too difficult, proper measures may be taken to explain these. (Dictionary work can be correlated in the upper grades. However, should the textbook problem be above the average child's head, other problems from the child's experience and within his mental grasp may be introduced in great variety.

III. *Failure to see all the elements of the problem.* Pupils often lack the ability to retain elements of a problem in mind while attending to new elements.

Remedies: If a pupil is to learn the secret of solving a large majority of the problems offered for solution, he must become acquainted with the natural steps of procedure, and endeavor to use them. He must be encouraged to start on a problem and think it through without any hesitancy. What steps, then, should a child consciously take when confronted by a problem?

(1) Read the problem thoroughly, looking up any strange or unfamiliar words.

(2) Determine what is given.

(4) Determine the process or processes to be used.

(5) Estimate the result.

(6) Statement and solution of the problem.

(7) Checking results. Possibly a good plan is to have the children write for every problem the steps given. Find process. This leads not only to thought and to a familiarity with the conditions and elements in the problem, but also to a recognition of the goal of the problem.

IV. *Failure to understand fundamental number relations.* Often children are taught each unit of arithmetic in isolation from all other units. Each fact or process is crowded separately into the child's mind, no association being made between division and multiplication, or multiplication and division, or subtraction and addition, etc. Given a jumbled mass of facts, with no apparent relation, the child's mind suffers from arrested development.

Remedies: There is a great necessity of keeping a fact and its "cousins," or related facts, associated by developing and drilling upon them as one family.

When the children in the upper grades are learning how to find area of rectangles, besides knowing that $A=L\times U$, they should also know that $L=A\div U$, and $U=A\div L$.

To stress these relations the teacher may give simple oral problems, associating original problems with their two "cousins" until the children see the relationshi p between problems as clearly as they see relationship between number facts. For example:

(1) The length of a rectangle is 9 feet and the width 6 feet. Find the area.

(2) The area of a rectangle is 54 square feet and the width 6 feet. Find the length.

(3) The area of a rectangle is 54 square feet and the length 9 feet. Find the width.

When more difficult problems are introduced, the children might be encouraged to supply the "two cousin problems" in addition to the first original one.

V. *Failure to transfer knowledge in simple oral problems to more difficult written problems.*

Do pupils who can solve problems when the numerical work

is simple enough to be done mentally fail to carry over these abilities in working similar problems which require paper and pencil?

For the purpose of answering this question, a test was devised. There were three sets to the test; each set involving (1) the written process alone, (2) a problem simple enough to be done mentally, (3) a similar problem with numbers large enough to require the written process.

The results indicate that the pupils' inability to carry over a process they know to a problem requiring written work is a frequent difficulty.

Remedies: Aside from the fact that the processes should be drilled upon until they become automatic, frequent drills in simple problems followed by similar but more difficult ones at the beginning of a problem-solving period, may help.

VI. *Lack of knowledge of fundamental facts and concepts.* Sometimes a pupil has the correct picture of the situation described in a problem, but lacks a knowledge of the facts upon which the solution depends. For example: "The cross-section of a railroad tunnel 132 yards long is in the form of a 16 foot square surmounted by a semi-circle. Find the number of cubic yards of earth removed in its excavation." The pupil sketched the picture very accurately. She stated that the problem was to find the volume of a prism and half that of a cylinder and add them. She then failed in the solution because she tried to find the volume by multiplying the perimeter of the cross section by the length of the tunnel.

Remedies: Often this difficulty is due to the skipping of grades or double promotions whereby the child has lost some necessary information. There is nothing to do but re-teach these fundamental rules and again present them as concretely as possible.

VII. *Failure to resist the disturbance caused by pre-conceived ideas.* "To prevent is easier than to cure" is true not only in the medical world, but also in our own arithmetical world. Pupils sometimes get an erroneous idea in their early school lives and carry that idea all the way up through the grades. For example: "James weighed 85 lb. a year ago. He has gained

12 lb. The gain is what decimal part of last year's weight?"

Teacher: Why did you multiply?
Pupil: Because the problem says "of."
Teacher: Do you always multiply when the problem says "of"?
Pupil: Yes.
Teacher: Why?
Pupil: Because "of" means to multiply.

Remedies: The teacher's task in blotting or erasing that idea from the child's mind is a most difficult one. Usually the young mind retains what has been impressed upon it.

VIII. *Failure to interpret cues correctly.* In an effort to help the pupil, often confusion results instead of enlightenment. For example: "Ralph is making a coat rack to contain 6 hooks at equal distances apart. It is to be 4 feet 2 inches from the first hook to the last. How far apart must they be placed?" (By a diagram of the coat rack, show that 5, and not the number of hooks, is the divisor.)

Teacher: Why is your diagram puzzling you?
Pupil: I do not know whether to put 6 hooks or 5 hooks?
Teacher: What does your problem say?
Pupil: 6 hooks but it says to divide by 5.

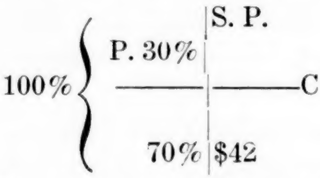
Remedies: This proves that in spite of the help given in the problem, pupils should not be permitted to attack the problem without a preliminary explanation (especially on the drawing) by the teacher. Occasionally, simple problems containing cues or suggestions might be given to the pupils.

IX. *Inability to read between the lines.* Pupils who can solve problems when all facts are given and when these are comparatively simple, fail to interpret the meaning of problems when all of the facts are not given. For example: "One year Chicago won 100 American League games and lost 54. What was her per cent?" (In majority of cases 100 becomes the divisor and 54 the dividend.)

Remedies: This type of problem calls in to play the child's power of reasoning and detection. Some such problems might prove very valuable for practice.

X. *Unfamiliarity with problems the condition of which are difficult to imagine.* In some problems (like those involving per cents and fractions) the situation may be clear and life-like, the wording of a comprehensive nature, the processes known by the child, etc. Yet the conditions may be difficult to imagine. For example: (1) A dealer paid $42 for a chair. For what must he sell it to make a gross profit of 30% of the selling price?

Remedy: Using the graph to picture of the problem is shown to the child. Thus,



$$70\% \times \text{S. P.} = \text{C.}$$
$$70\% \times \text{S. P.} = \$42.$$
$$\$42 \div 70\% = \text{S. P.}$$
$$\$60 = \text{S. P.}$$

(2) A boy had 24 marbles left after losing 1/5 of what he had. How many had he at first?

At first.

1/5 6 m.	1/5 6 m.	1/5 6 m.	1/5 6 m.	1/5 6 m.

Left.

X. *Lack of reasoning power.* An experiment was conducted, the results of which proved that a large percentage of Seventh Grade pupils that were tested, do not reason in attempting to solve arithmetic problems. Many of them appear to perform almost random calculations upon the numbers given. When they do solve a problem correctly, the response seems to be determined largely by habits. If the problem is expressed in unfamiliar terminology or if it is a "new" one, relatively few pupils appear to attempt to reason. There is also a general tendency among pupils to try to remember what someone else has said or to try to say what they think the teacher wants them to say, rather than "thinking the thing through" and reasoning it out for themselves.

Remedies: (1) Introducing a variety of problems. Have problems on cards; have problems written on the board; read problems from books; let children make up problems.

(2) Give different types of problems.

a. Problems without numbers. For example: George earns a certain amount of money each week. He paid a certain amount down on a bicycle. If he pays 1/4 of his earnings each week on the balance due on the bicycle, how can he find how many weeks it will take him to pay for the bicycle?"

b. Problems in which children supply the question. For example: "A school with 850 pupils on roll had an attendance of 825 pupils one day." Child may ask any of the following questions: "How many stayed at home? What part stayed at home? What per cent stayed at home? What part attended? How many more attended that stayed at home, etc?"

c. Problems in which children approximate results. For examples: "Gasoline is $0.27½ per gallon. Mr. Allen bought 5 gallons. About how much did it cost him?" Child says, "5×$0.27=$1.35." He then steps to the board and works it out to see how near approximation is to the exact answer—$1.37½.

d. Problems is which some facts necessary for solution are omitted and must be supplied; in other words, incomplete problems. For example: "John bought a radio set for $85. He sold it to his brother at a profit. Did he get enough money from the sale to buy a baseball outfit?"

(3) More oral work in problem solving than written work. During the oral lesson, the child is under direct guidance of the teacher. The written lesson should be used for the purpose of finding out whether all of the children have grasped the work that has been previously taught in oral lessons.

(4) Frequent use of thought questions. For example: "Knowing that 12 lbs. of butter cost $5.76, why do we divide the cost by 12 to get the price of 1 lb?"

XII. *Lack of initiative and self-confidence.*

Remedies: Discuss privately easier problems the solutions of which he already knows; show him frequently how to make a study attack; furnish him with supplementary equipment and books to arouse interest; let the pupil often submit written work for approval.

As the ultimate purpose of all arithmetical study is the

solving of problems and as so many difficulties arise in the reasoning of these problems, the teacher's aim, in choosing and using these problems, will be to have the child gain power to analyze quantitative situations and to look out upon his environment with mathematically educated eyes.

ALGEBRA AND THE SEVEN OBJECTIVES

By LORRAINE SHELL,
Okolona, Mississippi

The place that algebra holds in the curriculum of the High School can be justified by the contributions it makes indirectly and directly to three cardinal principles of education: Citizenship, Character, and Vocation.

For analysis the values commonly claimed for the study of algebra may be considered under two general headings: (1), those values which arise from the relatively direct and specific uses of algebra, (2), those values which may arise indirectly through the development of mathematical concept or through the transfer of improved efficiency.

Twenty-five years ago algebra was usually taught as though it were a purely mathematical discipline, unrelated to life—a mere meaningless puzzle. The purpose in its teaching sems to have been to make mathematicians. Now, we have a clearer vision of the value of the subject, for we are at present concerned with establishing objectives. Now our purpose is to develop well-informed American citizens. Therefore, we say algebra should be studied because general information requires it. If algebra did not touch many great business enterprises, all kinds of engineering, all our conception of the infinite about us, many great industries, the work of the twentieth century artisan, navigation and the building of aeroplanes, then it's study might be considered a luxury for the scholar alone. But algebra does touch all these lines of mental and manual activity, and hence it is a subject about which every person should be somewhat informed.

Today a person would feel his ignorance if he had to look with lacklustre eyes upon a simple formula such as may be found in

the popular scientific journals, in an everyday article on astronomy, in a boy's manual on the airplane or radio, or in any one of hundreds of articles in our popular encyclopedias. Today there is no artisan who takes a trade journal or who reads a manual devoted to his line of work who does not meet a formula language and who does not need to know how to evaluate and manipulate it. If he knows how to use the formula, he will need to know also how to solve the simple equation, and to know the universal language of algebra. Take for instance the law of falling bodies. Through this law, which expresses concrete facts of daily experience, the pupil learns the use and meaning of formulas and sees that law and order exist in the concrete world of material experience. He finds that a law once learned can be translated into a formula. He learns to seek for significiance and organization—in other words, to generalize. Every person must also know the meaning of a simple graph and of negative numbers or else feel the stigma of ignorance of the common things about which the educated world talks and reads.

Every one should be able to understand and interpret correctly graphical representations of various kinds such as nowadays abound in popular discussions of current scientific, social, industrial and political problems. This applies to the representation of the statistical data which is becoming increasingly important in the consideration of our daily problems, as well as to the representation and understanding of various sorts of dependence of one variabie quantity upon another. By the use of graphs the facts represented may be learned and grasped much more rapidly and in a more accurate and truthful form. Such reading is also often not only an economy to the mental powers of assimilation, but also a postive stimulus to the imagination and the reasoning powers. Graphs are especially helpful in aiding managers of large corporations to make quick and sound decisions where speed is of vital importance and to help corporations to keep in close and sympathetic contact with its employees, stockholders, and the public, thus making the business a corporative concern in a large way. If properly presented and developed, graphs have equally important disciplinary and cultural values. The study of them affords training in system and

organization, shows the relations of the component parts of a whole, develops the pupil's power of initiative, invention, and resourcefulness, or what we generally call self-activity, develops a certain amount of original thought, power, and reasoning in a pupil and develops the ability to select the most important fact learned. It is plain, therefore, that this topic trains the youthful mind to reason, to infer, to deduce, and to imagine in vitally useful ways. These valuable traits will transfer to other situations in life if the teacher holds them up to the pupils as ideals.

The negative number may be made to function in every-day life problems if the child is made to see that it may be used to indicate subtraction, to indicate a "shortage" or the mean "in the opposite direction". When a pupil comes to 3—8, he needs to know the second meaning. The result is "five in the hole", a "shortage" of five, or —5. The minus sign now indicates a suspended subtraction which we are unable as yet to perform, a condition which might unfortunately happen to one's bank account. In the scoring of games, children become familiar with the negative number. It is an idea which merchants might and do use in marking a bale of silk or other goods which was found to be of short weight.

Algebra exemplifies most typically, clearly and simply certain modes of thought which are of the utmost importance to every one. Its aim is to train the students in correct and useful habits of cogent thinking—to encourage them to face problems sanely and rationally. It is a requisite for success in any occupation to be able to grasp a situation, to seize the facts, and to perceive correctly the state of affairs.

Let us consider what connection algebra can show between good mental habits and good citizenship. It is generally accepted by authorities in this work that the mental habits of neatness, orderliness, accuracy, persistence and attention result from the successful study of algebra. The question is: Can these traits be generalized, divorced from mathematical content and utilized in their efficiency for other contexts and other situation in life? Yes, they can be, if they are in the hands of a worthy teacher who will hold them up as ideals. Through correct teaching procedure the mental habits of neatness can be made to con-

tribute to personal neatness and civic pride. It seems that orderliness might be properly joined with neatness as their civic manifestations are identical.

The community whose citizens possess accurate minds is indeed fortunate, for, in that community, guesses will never be advanced as to the cost of some project for the likelihood of passing a desired piece of legislation. Only facts that have been determined by experts to be correct could be admitted to public discussions. Again the faculty of persistence can not but augment the civic good, and in what other subjects do we have the fact brought home so squarely that to achieve we must work? There is no doubt that the importance of this realization by the citizens cannot be underestimated.

The next mental habit—that of attention—is one of the most important. The novice soon sees that he can attain to very little mathematical learning unless he gives it his undivided attention. If it be remembered that in the average community less than ten per cent of the voters dictate the policy, one can not fail to appreciate the evils of inattention in civic progress. If it be remembered that the time-honored and proven saying "A thing worth doing is worth doing well", depends squarely on the principle of attention, a small part of the value of this mental habit is seen.

In further considering the social-civic values of algebra, we find there is a certain intellectual honesty developed. This trait results, if properly taught, from the consideration in problems of that which is essential and that which is irrelevant. You will find the mathematically-trained citizen perhaps annoying you by his persistent "How do you know?", but you must concede that he will not admit into the discussion any fanciful irrelevant padding. When one realizes how much he must learn to get all the facts, he is not apt to waste time with anything off the subject. In other words, he becomes mentally efficient.

If the subject is judiciously taught, algebra shows that it concerns itself not only with proving that a statement is true, but in discovering truth for ourselves. It gives the student ideals of exact methods of procedure in getting at a truth-ideals which can then be generalized to any extent one desires. The

pupil who discovers that one equation really has a root and that it obeys the laws of other equations has made a step toward devoting himself to a search for truth. He has developed the ideal of devotion to truth and thereby is contributing to the upbuilding of character. The pupil discovers in connection with the algebraic equation that there are eternal verities. This means that the correct statement that an equation has a root is true "yesterday, today and forever". The square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs" is a verity which is true independently of time and place and therefore cannot be questioned. A pupil learns then that a human law may change, every building must sometimes decay, all life is continually changing, but that $(a+b)^2$ is always equal to $a^2+2ab+b^2$. Concrete cases of such everlasting truths should make an appeal to the pupil, and they will if the pupil is made conscious of this as an ideal independent of context. Especially are these ideals needed among the more advanced pupils in colleges and universities in view of the extraordinary changes now taking place in general-science theory.

From the study of algebra, the pupil also learns to feel a responsibility for the correctness of a result; thus developing self-confidence. He realizes that no problem should be considered solved without previously making a common-sense estimate of the probable result. That transfer can further take place is shown from the fact that one can be made to feel satisfied with only thorough work and precision of statement and dissatisfied with vague results. He also exercises other mental traits in developing the power of concentration, the capacity to generalize conceptions, the habit of self-scrutiny and the power of self-reliance in attacking all problems. These, as ideals reacting to different situations, will not only make a better citizen but will at the same time help to build up character.

Algebra may justify itself as educational material aside from its applied values. For pupils who expect later to become mathematical specialists, such engineers, astronomers and inventors, or who may have opportunities to employ mathematical facts, principles or processes extensively in advanced work, systematic and logically-organized courses in algebra are justi-

fied on the basis of direct values. This conclusion is reached on the ground that algebra is organized for the purpose of giving the pupil credit for admission to college, thus making it possible for him to secure a higher education and prepare for a vocation that requires greater basic educational preparation. Since it is directly vocational only for the small per cent who continue their academic education, it should be systematically studied only by those who are mathematically inclined and mathematically capable.

Surely the subject of algebra is such as to contribute to the preparation of the young person to become a civic, social and economic asset to the community and to himself. It arouses and develops latent interest and power. Moreover, it is not without its cultural value. The subject matter is rearranged and the method of presentation is such that it aims throughout to develop understanding and mental power. It also prepares the student who loves mathematics to secure a foundation for a continued study in order that he may apply his knowledge in some vocation. These are indeed some worthy criteria for justifying the place of algebra in the schools of today.

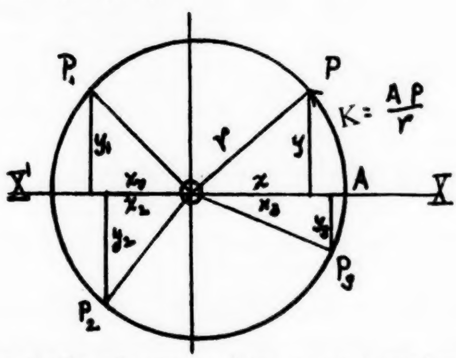
TRIGONOMETRY WITHOUT ANGLES

By W. PAUL WEBBER

Much has been said about the pedagogical value of the historical development of a subject. It has been said that when the historical development can be used without loss of time and without confusion of ideas, it is to be preferred to some other method. There are notable cases where the historical development cannot be advantageously used in presenting the subject. One such is logarithms. The commonly accepted presentation does not event suggest the historical origin of the idea. Recently the definitions of elementary trigonometry have come up for discussion. There are persons who cannot be reconciled to the notion of an angle of magnitude over 180°. There are others who feel that it is quite as desirable to make addition of angles apply to all cases, and consequently to admit angles of

"all magnitudes." It is not the purpose of this paper to decide this question, which is still largely a matter of opinion, but rather to show how the definitions of trigonometry may be obtained without the use of the term angle just as well as with it. In fact it will make no difference. It is to be remembered that probably the world would have waited longer to get these definitions without the term angle than with it. But once the idea is in mind, the theory can be developed either with or without the angle.

It is quite a mater of convention that we lay off the number scale on a straight line. For many purposes this is convenient. It is quite as easy to lay off a number scale on the circumference of a circle of any radius, r. In this case the same point on the circumference will represent many numbers, instead of one only. This has some advantages, for it opens up a new field of functional relations, namely, periodic functions in the realm of the real numbers.



Consider a circle of any radius, r, and center O. A circle of unit radius is convenient but not necessary for our purpose. Let X'CX be a fixed line, and A the right hand point where the circle intersects the line X'CX. The point A will be taken as the zero point of the scale. Now with r as the unit of measure lay off the scale around the circumference. It can be laid off in two opposite senses. The counter clock-wise sense will be called the positive sense. Then clock-wise will be negative.

If some variable, say K, is taken to represent numbers on the scale, it is seen that every value of K can be represented by a point (number) of the scale. For, suppose a flexible tape line

is fixed with its zero at A. It is only necessary to find the value of K on the tape line, and then wind around the circle until the point on the tape line comes into contact with the circumference. The point thus obtained will represent the given value of K. It is true also that every point on the circumference must represent an indefinite number of different value of K. These values will differ successively by

$$\frac{2\pi r}{r}=2\pi.$$

Now it is to be remembered that each point on the circumference has a unique pair of rectangular coordinates of which the line X'OX is the axis of abscissas. If these coordinates are called x, y, there is one pair x, y corresponding to each point and one point corresponding to each pair x, y. The value of r is, of course, the same for all points on the circumference. Further, corresponding to each point, and consequently to each value of K, is one, and only one, right triangle whose legs are x, y, and whose hypothenuse is r, so that for every point P and for every value of K the relation $x^2+y^2=r^2$ holds. Further it is now easy to see that for each value of K and for each point P, the ratios y/r, x/r, y/x are uniquely determined. This correspondence indicates a functional relation between K as independent variable, and each of these ratios as dependent variables. To conform to the known definitions of trigonometry we may now name these relations as

Sin K$=y/r$, Cos K$=x/r$, and Tan K$=y/x$, respectively.

The other three functions may be defined as reciprocals, respectively, of these. Thus all the trigonometric functions have been defined without using the term angle. The defiinitions may be regarded as the result of laying off the number scale on a circle instead of on a straight line, and setting up correspondences between these numbers and the rectangular coordinates of the points representing the numbers of scale. From the analytical viewpoint the application to angles is merely a special case.

To obtain the fundamental identities it is only necessary to observe that for all positions of P, and for all values of K, the relations, $x^2+y^2=r^2$ holds, so that, no matter what value K has,

$$(y/r)^2+(x/r)^2=\frac{x^2+y^2}{r^2}=r^2/r^2=1$$

The derivation of other identities is now easy.

The periodic property of these functions is now obvious. For, if K_1 is any given value of K, and x_1, y_1 the coordinates of the corresponding point P on the circumference, then

$$\text{Sin } K_1=y_1/r, \text{ Cos } K_1=x_1/r, \text{ Tan } K_1=y_1/x_1.$$

Then, also, will

Sin $(K_1+2\pi)=y_1/r$, Cos $(K_1+2\pi)=x_1/r$, Tan $(K+2\pi)=y_1/x_1$, etc. It appears that 2π is a period for all the trigonometric functions. It will be remembered that 2π is equialent of 360°.

Enough has been said to show how to obtain the definitions at once for any and all values of K. The discussion of values at the quadrant limits, fundamental regions, etc., may be taken up in the usual way. It also appears from what has been said that the inverse relations, Arc sin K, Arc cos K, Arc tan K are not single valued, but are many valued. That is, to any given value of one of the functions there will correspond indefinitely many values of the variable K, that is of the angle.

To be sure that we are permitted to use the term angle with our definitions it is only necessary to recall that geometry teaches that angles at the center of a circle are measured by their intercepted arcs.

SEXTANT AND BI-SEXTANT TRIANGLES

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1. Their Geometry

(1) *Definitions.* A Sextant or Sextantal Triangle is a plane triangle having one of its angles equal to Sixty Degrees. A Bi-sextant or Bi-sextantal Triangle is a plane triangle having one of its angles equal to One Hundred and Twenty Degrees, or twice Sixty. As associated with these triangles we shall refer to the Right Triangle as Quadrant or Quadrantal Triangle.

(2) *The Right Triangle Supreme.* In the commonwealth of

Polygonia the right triangle, Orthogon, occupies a place of preeminence—due, shall we say, to his outstanding quality of uprightness; a desideratum devoutly to be desired of all who aspire to positions of prominence in whatever state, whether geometrical or geographical. Orthogon is at once the gatekeeper and the referee of the commonwealth, for it is through him that we are introduced to the other inhabitants of this geometrical state, and any investigation of any citizen thereof is ultimately referred to him.

This honor bestowed upon the right triangle—and he is every whit worthy of it—is at the hand of mankind. If, however, we had learned our geometry from Mother Nature things might have been somewhat different, as we shall presently try to show.

(3) *A Triangle Triumvirate.* It is not our purpose in this paper to detract anything from the honor and prestige of the right triangle, Orthogon, but to call attention to two triangles closely associated with him and standing, so to speak, on his right hand and on his left hand—a sort of Aaron and Hur holding up the hands of a geometrical Moses.

The two triangles just referred to are those defined in the first paragraph. The Equilateral Triangle, which is also equiangular, is a special Sextantal Triangle. In speaking of the equilateral triangle in the class of regular polygons we shall refer to it as Sextantal. In the same category we shall refer to the square as Quadrantal, and the hexagon as Bistantal.

There are only three plane-filling regular polygons and they are the three just referred to. It is the study of the triangles each bearing an angle of these three plane-filling polygons that shall occupy our attention in this paper.

(4) *The Geometry of Mother Nature.* We have previously referred to the geometry exhibited by Mother Nature. She is very careful to avoid partiality. Save (possibly) the single tendency of things to grow "straight up," nature has not singled out the right angle above any other angle. For indeed, while the stem or trunk in vegetation is prone to grow "straight up" in surprisingly few cases do the leaves and branches tend to grow "straight out" from the stem or trunk. The prominence of the other two angles we are considering,

namely sixty and one hundred and twenty degrees, is familiar to all. Witness: the honey-bee cell, ice crystals, and in fact all forms of crystaline composition, the distribution of leaves and petals in flowers, etc.*

(5) *Some Properties of Sextantal and Bisextantal Triangles.* Enough of this sparring. Let us come to grips with our subject. The angles we are considering are 60°, 90°, 120°. They are the angles of a regular triangle, a regular quadrangle, and a regular hexagon. They are multiples of 30°, 2, 3, 4. Hence they are in an arithmetical progression with common difference 30. They are submultiples of a straight angle or 180°, 1/3, 1/2, 2/3; hence, of a perigon or 360°, 1/6, 1/4, 1/3.

In the triangle ABC we shall take the angle C as 60°, 90°, or 120° as the case may be. The following shall be the order of the sides:

For the Sextantal Triangle, $A \leqq C \leqq B$.

For the Quadrantal and Bisextantal Triangles, $A \leqq B \leqq C$.

We state the following properties for the most part without giving the proofs.

1. The sextantal triangle which is also quadrantal, sexto-quadrantal, is the well-known 30°-60°-90° triangle.

1′. The bisextantal triangle can be neither sextantal nor quadrantal.

II. The isoscles sextantal triangle is easily seen to be equilateral and equiangular, that is triply-sextantal. Such a triangle is bisected into two sexto-quadrantal triangles by any altitude.

II′. The isosceles bisextantal triangle is bisected into two sextoquadrantal triangles by the altitude upon c, the side opposite the bi-sextantal angle.

III. Any sextantal triangle may be dissected into a triply-sextantal triangle and a bisextantal triangle by a line from B cutting off on b a segment equal to a measured from C. Any sextantal triangle is dissected into a sexto-quadrantal triangle and a quadrantal triangle by the altitude on b.

III′. Any bisextantal triangle is dissected into two sextantal triangles by the bisector of the angle C.

*An interesting study has been made along this line by Miss Marie Gugle, well-remembered by many of the readers of this magazine, under the title Dynamic Symmetry.

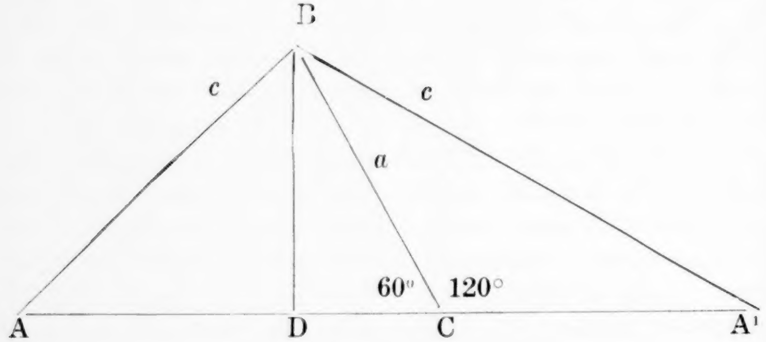

Figure 1.

In Figure 1 let BD be the altitude of the two triangles, the one, ABC, being sextantal, the other, A'BC, being bisextantal: Then BD$=(a/2)\sqrt{3}$ and CD$=(1/2)a$. For the sextantal triangle,

$$c^2=3c/4+(b-(1/2)a)^2=a^2+b^2-ab$$

For the bisextantal triangle,

$$c^2=3a/4+(b+(1/2)a)^2=a^2+b^2+ab$$

Some further relations we now exhibit by parallel columns using the customary notation for triangles: Area $\triangle$, Perimeter 2s, Circumradius R, Inradius r.

	Sextantal Triangle S	Quadrantal Triangle Q	Bisectantal Triangle B
C=	60°	90°	120°
A+B=	2C	C	(1/2)C
$c^2=$	a^2+b^2-ab	a^2+b^2	a^2+b^2+ab
$s(s-c)=$	$(3/4)ab$	$(2/4)ab$	$(1/4)ab$
$\triangle=$	$(1/2)ab\sqrt{3}$	$(1/2)ab$	$(1/2)ab\sqrt{3}$
$R^2=$	$(1/3)c^2$	$(1/4)c^2$	$(1/3)c^2$

$r^2=$ $\left\{\begin{array}{l} (1/4)d^2 \quad \ldots\ldots \quad (2/4)d^2 \quad \ldots\ldots \quad (3/4)d^2 \\ \text{where } d \text{ is the distance from C to the incenter.} \\ (1/3)(s-c)^2 \quad \ldots\ldots \quad (s-c)^2 \quad \ldots\ldots \quad 3(s-c)^2 \end{array}\right.$

In this and the following paragraph we shall continue our comparison of the three triangles by equating certain parts and superposing the triangles on these common parts. In this section we shall take the angle A and the side b as common to the three triangles. Thus, in Figure 2 ACB$_1$ is the sextantal tri-

angle, ACB_2 is the quadrantal triangle, and ACB_3 is the bisextantal triangle. We shall call AB_1 c_1, AB_2 c_2, AB_3 c_3, and CB_1 a_1, CB_2 a_2, CB_3 a_3.

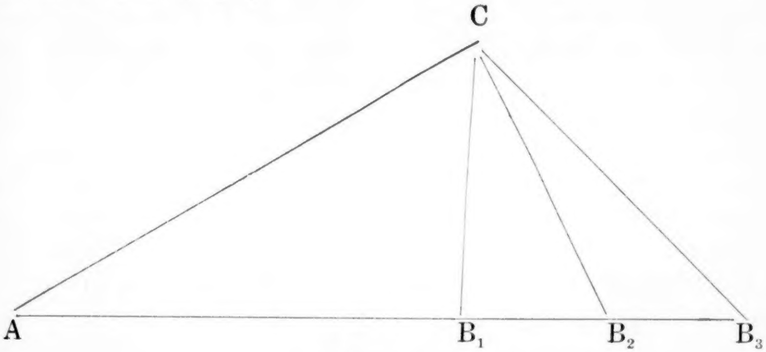

Figure 2.

The sides CB_2 and CA bisect the interior and exterior angles B_1CB_3, hence A and B_2 are harmonic conjugates with respect to B_1 and B_3 and vice versa. Hence the hypotenuse of the quadrantal triangle, c_2, is the harmonic mean of c_1 and c_2, the corresponding sides of the sextantal triangle and the bisextantal triangle. Symbolically

$$2/c_2 = 1/c_1 + 1/c_3.$$

A bit of reduction will show that the sides a also contain this same relation. Furthermore the three areas are connected by this same relation. Since $c_1 = R_1\sqrt{3}$, $c_2 = 2R_2$, $c_3 = R_3\sqrt{3}$, the R's are seen to be connected by the relation $\sqrt{3}/R_2 = 1/R_1 + 1/R_3$.

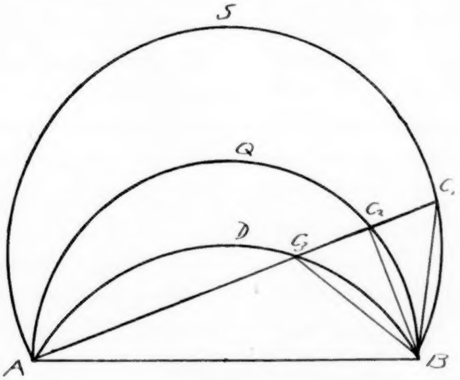

Figure 3.

In this final paragraph the circular arcs ASB, AQB, ADB constructed on the common chord AB measure 240°, 180° and 120° respectively, hence angles having their vertices on these arcs and their sides passing through A and B will be sextantal, quadrantal, and bisextantal. Drawing any line through A cutting these arcs in C_1, C_2, C_3 and drawing BC_1 BC_2, BC_3 we have a sextantal, a quadrantal, and a bisextantal triangle. They have the side $AB=c$ and the angle A in common. Let $BC_1=a$ $BC_2=h$, $BC_3=a$, $AC_1=b_1$, $AC_2=b$, $AC_3=b_3$. Noting that BC_1C_3 is triply sextantal we see that b_1 b, b_3 are in an arithmetical progression, the common difference being $(1/2)a$. Hence the areas are in an arithmetical progression the common difference being $(1/2)(BC_1C_3)$.

The three areas therefore are

$$(h/6)(3b+h\sqrt{3}), \qquad (h/6)(3b), \qquad (h/6)(3b-h\sqrt{3})$$

Let the line AC revolve about A. The bisextantal triangle has its maximum area when AC passes through D when $h=(1/2)c$. At this juncture the three triangles have areas

$$(2/3)hb \qquad (1/2)hb, \qquad (1/3)hb.$$

or

$$(c^2/6)\sqrt{3}, \qquad (c^2/8)\sqrt{3}, \qquad (c^2/12)\sqrt{3}.$$

The quadrantal triangle has its maximum when AC passes through Q at which time $h=b$ and its area is $(\frac{1}{2})h^2$. The three areas now being:

$$(h^2/6)(3+\sqrt{3}), \qquad (h^2/6)(3), \qquad (h^2/6)(3-\sqrt{3})$$

since $2h^2=c^2$ these forms in terms of the common side c are

$$(c^2/12)(3+\sqrt{3}), \qquad (c^2/12)(3) \qquad (c^2/12)(3-\sqrt{3})$$

The sextantal triangle has its maximum area when AC passes through S. The areas of the three triangles are

$$(c^2/4)\sqrt{3}, \qquad (c^2/8)\sqrt{3}, \qquad \text{Nil.}$$

The maximum areas for the three triangles, namely, $(c^2/4)\sqrt{3}$, $c^2/4$, and $(c/12)\sqrt{3}$ form a geometrical progression with common ratio $1/\sqrt{3}$, that is, the maximum area of the quadrantal triangle is a mean proportional between the maximum areas of the sextantal and bisextantal triangles having a common side, opposite the naming angles and a common angle.

We have but scraped the surface of the interesting relationships of these three kindred triangles. I trust that many more will be forthcoming in the near future.